Fixed Point Theorems

We prove some fixed point theorems without use of homotopy. We follow 'Banach Space Theory, by Fabian et al for Brouwer's Fixed Point Theorem, Linear Operators by Dunford and Schwartz, Part I for Theorem 4 and 'A course on Functional Analysis by Conway for the others . We give more detailed proofs than the one in above books with the hope that this will make it easier for students to understand the proofs.

Theorem 1 [Brouwer's Fixed Point Theorem]
Any continuous map of $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ into itself has a fixed point.

A proof of Brouwer's fixed point theorem:

Lemma 1

If $f : S^{n-1} \to \mathbb{R}^n$ is continuous, $< f(x), x >= 0$ and $f(x) \neq 0$ for all x then there exists a continuously differentiable function $g : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$ such that $\|g(x)\| = 1$ for all $x \in S^{n-1}, g(rx) = rg(x)$ for all $x \in \mathbb{R}^n \backslash \{0\}$ for all $r > 0$ and $< g(x), x >= 0$ for all $x \in S^{n-1}$.

Proof: Let $m = \inf\{\|f(x)\| : x \in S^{n-1}\}$. Of course, $m > 0$. There exists a polynomial $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|p(x) - f(x)\| < m/2$ for all $x \in S^{n-1}$. [If two polynomials coincide on S^{n-1} then their difference has infinitely many zeros, hence it is 0. Thus we can apply Stone -Weierstrass Theorem]. Define $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(x) = p(x) - < p(x), x > x$. Claim: h is 'smooth', $< h(x), x >= 0$ whenever $\|x\| = 1, \|p(x) - h(x)\| < m/2$ for $x \in S^{n-1}$. Only the last property needs a proof. $\|p(x) - h(x)\| = |< p(x), x >| = |< p(x) - f(x), x >| \leq \|p(x) - f(x)\| < m/2$ so the claim is proved. Claim: $h(x) \neq 0$ if $x \in S^{n-1}$. Indeed $h(x) = 0$ implies $\|p(x)\| < m/2$ and $\|f(x)\| < \|p(x)\| + m/2 < m$ contradicting the definition of m. Now let $g(x) = \frac{\|x\| h(x/\|x\|)}{\|h(x/\|x\|)\|}$ for $x \in \mathbb{R}^n \backslash \{0\}$. We have $< h(x/\|x\|), x >=< h(x), x >= 0$ for $x \in S^{n-1}$ and the proof is complete.

Lemma 2

Let $K \subseteq \mathbb{R}^n$ be a non-void compact set. Let $f : \Omega \to \mathbb{R}^n$ be continuously differentiable where Ω is open and $K \subseteq \Omega$. Let $f_t(x) = x + tf(x)$ for $x \in \Omega$. Then, for $|t|$ sufficiently small, f_t is one-to-one on K and $m_n(f_t(K))$ is a polynomial in t.

Claim: there exists $c \in (0, \infty)$ such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for all $x, y \in K$. If not, then there exist $x_n, y_n \in K$ such that $\|f(x_n) - f(y_n)\| > n\|x_n - y_n\| \, \forall n$. By going to a subsequence we may suppose $x_n \to x$ and $y_n \to y$. If $x \neq y$ we get $\|f(x_n) - f(y_n)\| \to \infty$, a contradiction. If $x = y$ then differentiability at x is contradicted. [In fact $\|f(x_n) - f(y_n)\| \leq \|f'(x)\| \|x_n - y_n\| + o(\|x_n - y_n\|)$]. Let $|t| < \frac{1}{c}$. Then $f_t(x) = f_t(y)$ implies $x - y = t[f(x) - f(y)]$

so $\|x-y\| \le |t|\,\|f(x)-f(y)\| \le |t|\,c\,\|x-y\|$ which implies $x=y$. The matrix $\left(\left(\frac{\partial (f_t)_i}{\partial x_j}\right)\right)$ is of the form $I+tM$ where $M=\left(\left(\frac{\partial f_i}{\partial x_j}\right)\right)$. Hence $\det\left(\left(\frac{\partial (f_t)_i}{\partial x_j}\right)\right)$ is a polynomial in t. Also $\det\left(\left(\frac{\partial (f_t)_i}{\partial x_j}\right)\right) = \det(I+tM) > 0$ for $|t|$ sufficiently small. Since $m_n(f_t(K)) = \int\limits_K \left|\det\left(\left(\frac{\partial (f_t)_i}{\partial x_j}\right)\right)\right| dx$ [see II, page 154 of Rudin's Real and Complex Analysis] we see that $m_n(f_t(K))$ is a polynomial in t.

Theorem 1 [Hairy Ball Theorem]

If n is an odd positive integer there is no continuous map $\phi : S^{n-1} \to \mathbb{R}^n$ such that $\phi(x) \neq 0$ for all x and $<\phi(x), x>=0$ for all x.

Remark: conclusion fails for $n=2$: $\phi(a,b)=(-b,a)$ has above properties.

Proof: suppose such a ϕ exists. By Lemma 1 there exists a continuously differentiable function $g : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n$ such that $\|g(x)\|=1$ for all $x \in S^{n-1}, g(rx)=rg(x)$ for all $x \in \mathbb{R}^n\backslash\{0\}$ for all $r>0$ and $<g(x),x>=0$ for all $x \in S^{n-1}$. Let $g_t(x)=x+tg(x), x \in \mathbb{R}^n\backslash\{0\}$. Let $K=\{x \in \mathbb{R}^n : a \le \|x\| \le b\}$ where $0<a<1<b$. For $|t|$ sufficiently small g_t is one-to-one on K and it maps K onto $\{x \in \mathbb{R}^n : a\sqrt{1+t^2} \le \|x\| \le b\sqrt{1+t^2}\}$. [Because $<g(x),x>=0$ for all $x \in S^{n-1}$ so $\|x+tg(x)\| = \|x+t\,\|x\|\,g(x/\|x\|)\| = \|x\|\,\|x/\|x\|+tg(x/\|x\|)\| = \|x\|\sqrt{1+t^2}$ by orthogonality]. It follows by Lemma 2 that $m_n\{x \in \mathbb{R}^n : a\sqrt{1+t^2} \le \|x\| \le b\sqrt{1+t^2}\}$ coincides with a polynomial in t for $|t|$ sufficiently small. However we can compute this measure explicitly and the value is $[b^n(1+t^2)^{n/2} - a^n(1+t^2)^{n/2}]m_n\{x : \|x\| \le 1\}$. Since $(1+t^2)^{n/2}$ is not a polynomial for n odd we have completed the proof. [Let $n=2m-1$. Suppose$(1+t^2)^{m-1/2}=p(t)$ for $|t|$ small, where p is a polynomial. Then $(1+z^2)^{2m-1}=p^2(z)$ for all $z \in \mathbb{C}$. The right side must be a polynomial with i and $-i$ as the only roots and since $p(t)$ is real for $|t|$ sufficiently small it follows that $(1+z^2)^{2m-1}=c(z-i)^k(z+i)^k$ for some k. We get a contradiction by comparing the degrees].

Theorem 2 [Brouwer's Fixed Point Theorem]
If $f : S^{n-1} \to S^{n-1}$ is continuous then f has at least one fixed point.

Proof: define $\phi : \mathbb{R}^n \to S^n$ by $\phi(x_1,x_2,...,x_n) = (\frac{2x_1}{1+\|x\|^2}, \frac{2x_2}{1+\|x\|^2}, ..., \frac{2x_n}{1+\|x\|^2}, \frac{\|x\|^2-1}{\|x\|^2+1})$. [It is trivial to check that ϕ does have its range inside S^n]. Let $\Phi(u) = \lim\limits_{t\to 0} \frac{\phi(x+tg(x))-\phi(x)}{t}$ where $u=\phi(x)$ and $g(x) = x - \frac{f(x)(1-\|x\|^2)}{1-<x,f(x)>}$. Let $(S^n)_l = \{x \in S^n : x_{n+1} \le 0\}$. Assuming that f has no fixed point we shall verify that Φ is a continuous map from $(S^n)_l$ into $\mathbb{R}^n$ such that $\Phi(x) \neq 0$ for all x and $<\Phi(u),u>=0$ for all $u \in S^n$. We will then show that there is a similar map on $(S^n)_u = \{x \in S^n : x_{n+1} \ge 0\}$ and that these two functions can be combined to give a similar function on the whole of S^n. This would contradict the Hairy Ball Theorem, thereby completing the proof of Brouwer's Theorem. Continuity of Φ is clear. Note that $\phi(x) \in (S^n)_l$ iff $\|x\| \leq 1$. Also note that if $u = \phi(x)$ then $\Phi(u) = \phi'(x)(g(x))$ so $< \Phi(u), u >=< \phi'(x)(g(x)), \phi(x) >$. Let $\|x\| = 1$. We have to show that $< \phi'(x)(g(x)), \phi(x) >= 0$. We show that $< \phi'(x)y, \phi(x) >= 0$ for any y. We have $\|\phi(.)\| \equiv 1$. Writing ϕ_i for the $i-th$ component of ϕ we have $\sum_{i=1}^{n+1} \phi_i^2(z) = 1$ for all z. Hence $\sum_{i=1}^{n+1} \phi_i(z) \frac{\partial \phi_i}{\partial x_j}(z) = 0$ for all z for all j. This gives $\sum_{j=1}^{n+1} y_j \sum_{i=1}^{n+1} \phi_i(z) \frac{\partial \phi_i}{\partial x_j}(z) = 0$ or $\sum_{i=1}^{n+1} (\sum_{j=1}^{n+1} \frac{\partial \phi_i}{\partial x_j}(z) y_j) \phi_i(z) = 0$. This says $< \phi'(z)y, \phi(z) >= 0$. This completes the proof of the fact that $< \Phi(u), u >= 0$ for all $u \in S^n$. It remains to show that $\phi'(x)(g(x)) \neq 0$ for any $x \in (S^n)_l$. For this we show that $g(x) \neq 0$ and that $\phi'(x)$ is one-to-one for each x with $\|x\| \leq 1$. Suppose $g(x) = 0$. Then $\frac{f(x)(1-\|x\|^2)}{1-<x,f(x)>} = x$. In particular $f(x) = cx$ for some scalar c. We have $cx(1 - \|x\|^2) = [1 - c\|x\|^2]x$ which gives $cx = x$. But then $f(x) = cx = x$ contrary to our assumption. Finally we show that $\phi'(x)$ is one-to-one for each x. Note that the range of ϕ is contained in $\{y \in \mathbb{R}^{n+1} : y_{n+1} < 1\}$. Define $F : E \equiv \{y \in \mathbb{R}^{n+1} : y_{n+1} < 1\} \to \mathbb{R}^n$ by $F(y_1, y_2, ..., y_{n+1}) = (\frac{y_1}{1-y_{n+1}}, \frac{y_2}{1-y_{n+1}}, ..., \frac{y_n}{1-y_{n+1}})$. F is a smooth function and a simple computation shows that $F(\phi(x)) = x$ for all $x \in \mathbb{R}^n$. It follows by Chain Rule that $F'(\phi(x))\phi'(x) = I$ for all $x \in \mathbb{R}^n$. This implies that $\phi'(x)$ is one-to-one.

This theorem does not hold in an infinite dimensional Hilbert space: if $f(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...)$ then f maps $\{x \in l^2 : \|x\| \leq 1\}$ into itself and is continuous. It has no fixed point.

Theorem 3

Any continuous map f of a compact convex K set in $\mathbb{R}^n$ into K has a fixed point.

Proof: assume first that $K \subseteq B \equiv \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Define $\phi : B \to K$ by taking $\phi(x)$ to be the unique point y of K such that $\|x - y\| \leq \|x - z\|$ for all $z \in K$. Such a vector y exists and is unique. Note that $\phi(x) = y = x$ if $x \in K$. Consider $f \circ \phi : B \to K$ as a function from B into itself. The function $g : B \to B$ defined by $g(x) = f(\phi(x))$ is continuous because ϕ is continuous: let $x_n \to x$. We have $\|x_n - \phi(x_n)\| \leq \|x_n - z\|$ for all $z \in K$; Hence, if y is any limit point of $\{\phi(x_n)\}$ then $\|x - y\| \leq \|x - z\|$ for all $z \in K$. This proves that $\phi(x)$ is the only limit point of $\{\phi(x_n)\}$ which lies in the compact set K. Hence $\phi(x_n) \to \phi(x)$. By Theorem 1 there exists $x \in B$ such that $f(\phi(x)) = x$. Since the range of f is contained in K we get $x \in K$. But then $\phi(x) = x$ so $f(x) = x$.

This proves the theorem when $K \subseteq B \equiv \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. For the general case choose R such that $K \subseteq \{x \in \mathbb{R}^n : \|x\| \leq R\}$. Let $K_1 = \{R^{-1}x : x \in K\}$. Then K_1 is a compact convex set and the function $f_1 : K_1 \to K_1$ defined by $f_1(x) = R^{-1}f(Rx)$ is continuous. By the first case there exists $x_1 \in K_1$ such that $R^{-1}f(Rx_1) = x_1$. If $x = Rx_1$ then $f(x) = x$.

Definition: a function $f : E \to X$ where X is a normed linear space (nls) and $E \subseteq X$ is called compact if $f(A)$ is relatively compact whenever $A \subseteq E$ is bounded.

Lemma 3

Let X be an nls and $K \subseteq X$ be compact. Let $\varepsilon > 0$ and $B(x_1, \varepsilon), B(x_2, \varepsilon), ..., B(x_N, \varepsilon)$ cover K where $\{x_1, x_2, ..., x_N\} \subseteq K$. let $m_i(x) = (\varepsilon - \|x - x_i\|)^+$ and $\phi(x) = \dfrac{\sum_{j=1}^{N} m_i(x)x_i}{\sum_{j=1}^{N} m_i(x)}$ for $x \in K$. Then ϕ is continuous on K and $\|\phi(x) - x\| < \varepsilon$ for all $x \in K$. Further $\phi(K) \subseteq co(K)$.

Proof: it is obvious that each m_i is continuous and $\sum_{j=1}^{N} m_i(x) > 0$ for all $x \in K$. Hence ϕ is continuous. If $x \in K$ then $m_i(x) \neq 0$ implies $\|x - x_i\| < \varepsilon$ and hence $\left\| \sum_{j=1}^{N} m_i(x)(x_i - x) \right\| < \varepsilon \sum_{j=1}^{N} m_i(x)$ which proves that $\|\phi(x) - x\| < \varepsilon$. [we have used the fact that $m_i(x) \neq 0$ for at least one i]. Last part is obvious.

Theorem 4 [Schauder Fixed Point Theorem]

Let E be a closed bounded convex set in an nls X and f be a continuous map of E into itself. If f is compact then it has a fixed point.

Proof: let $K = [f(E)]^-$. Then K is a compact subset of E. For each n let $\phi_n : K \to co(K) \subseteq E$ be a continuous function such that $\|\phi_n(x) - x\| < 1/n$ for all $x \in K$ for all n. This is possible by the previous lemma. Let $f_n = \phi_n \circ f$ so that f_n is a continuous map : $K \to E$. In the notation of previous lemma there is a finite set $\{x_1^{(n}, x_2^{(n}, ..., x_{N_n}^{(n}\}$ of K such that $\phi_n(K) \subseteq Y_n \equiv span(\{x_1^{(n}, x_2^{(n}, ..., x_{N_n}^{(n}\})$. Let $E_n = E \cap Y_n$. Then E_n is a compact convex set in the finite dimensional space Y_n. We claim that f_n maps E_n into itself. First note that $f(E_n) \subseteq f(E) \subseteq K$ so $f_n = \phi_n \circ f$ is defined on E_n . Also ϕ_n takes values in $co(\{x_1^{(n}, x_2^{(n}, ..., x_{N_n}^{(n}\})) \subseteq Y_n$ as well as in E so it takes values in E_n. By Theorem 2 there exists $y_n \in E_n$ such that $f_n(y_n) = y_n$. Since $y_n \in E$ and $f(y_n) \in K$ we have $\|\phi_n(f(y_n)) - f(y_n)\| < 1/n$ for all n. In other words $\|y_n - f(y_n)\| < 1/n$ for all n. Since $\{f(y_n)\} \subseteq K$ and K is compact there is a subsequence $\{f(y_{n_j})\}$ converging to some y. Now

$\|y_{n_j} - y\| \le \|f(y_{n_j}) - y\| + \|y_{n_j} - f(y_{n_j})\| < \|f(y_{n_j}) - y\| + 1/n_j \to 0$. This implies $f(y) = y$.

Theorem 5 [Schauder - Tychonoff FPT]

Any continuous map f from a compact convex subset K of a Hausdorff locally convex topological vector space X into K has a fixed point.

Preliminaries: we introduce an ordering for subsets of X^* as follows: $A \le B$ if for any $x^* \in A$ and $\varepsilon > 0$ there exists a finite subset $y_1^*, y_2^*, ..., y_k^*$ of B and $\delta > 0$ such that $x, y \in K$ and $|y_i^*(x) - y_i^*(y)| < \delta, 1 \le i \le k$ imply $|x^*(f(x)) - x^*(f(y))| < \varepsilon$. We observe that if $A \le B$ and $y^*(x) = y^*(y)$ for all $y^* \in B$ then $x^*(f(x)) = x^*(f(y))$. Claim: for any $x^* \in X^*$ there exists a countable family $B = \{y_1^*, y_2^*, ...\}$ such that $\{x^*\} \le B$. For this let $\varepsilon > 0$. First note that f is weak-weak continuous and K is compact convex in weak topology. By uniform continuity of $x^* \circ f$ on K with its weak topology $|x^*(f(x)) - x^*(f(y))| < \varepsilon$ if $x - y$ belongs to a suitable weak neighbourhood of 0. Hence there exists $y_1^*, y_2^*, ..., y_k^*$ and $\delta > 0$ such that $|y_i^*(x) - y_i^*(y)| < \delta, 1 \le i \le k$ implies $|x^*(f(x)) - x^*(f(y))| < \varepsilon$. Now vary ε over $\{\frac{1}{n} : n \ge 1\}$ to get a countable set $B \subseteq X^*$. For any $\varepsilon > 0$ choose n such that $\frac{1}{n} < \varepsilon$. There exist $y_1^*, y_2^*, ..., y_k^*$ and $\delta > 0$ such that $|y_i^*(x) - y_i^*(y)| < \delta, 1 \le i \le k$ implies $|x^*(f(x)) - x^*(f(y))| < \frac{1}{n} < \varepsilon$. It follows that if $|y^*(x) - y^*(y)| < \delta$ for all $y^* \in B$ then $|x^*(f(x)) - x^*(f(y))| < \varepsilon$. Hence $\{x^*\} \le B$. If we now repeat the argument for each element of B to get another countable set B_1, then repeat the argument for each element of B_1 and so on we end up with a countable family B_0 such that, together with x^* itself, we get a countable subset C of X^* which contains x^* with $C \le C$.

Lemma 4

Let K be a compact convex set in a locally convex Hausdorff topological vector space X. If K has at least two points and $f : K \to K$ is continuous then there is a proper subset K_1 of K such that $f(K_1) \subseteq K_1$ and K_1 is also compact and convex.

We first remark that **this lemma immediately yields Theorem 5**: there is a minimal non-empty compact convex set K_0 such that $f(K_0) \subseteq K_0$ and K_0 must be a singleton by the lemma.

Proof of the lemma: we first reduce the proof to the case when the topology of X is replaced by the weak topology. f is weak to weak continuous and K is weakly compact. If K_1 weakly compact, convex and contained in K then it is a weakly closed convex set, hence strongly closed. Hence it is a closed convex subset of K in the strong (i.e. original) topology, hence strongly compact also. Thus, we may and do assume that the topology of X is the weak topology. Now suppose $x, y \in K, x \neq y$. Choose x^*such that $x^*(x) \neq x^*(y)$. Let $B = \{x_0^* = x^*, x_1^*, ...\}$ be a countable subset of X^* containing x^* such that $B \le B$. Now $x_i^*(K)$ is compact for each $i \ge 0$. We may suppose $|x_i^*(z)| \le \frac{1}{i+1}$ for all i, for all $z \in K$. [This is because if $C = \{a_0 x^*, a_1 x_1^*, ...\}$ with each $a_i > 0$

then $C \leq C$]. Define $h : K \to l^2$ by $h(z) = \{x_i^*(z)\}$. h is continuous (by Dominated Convergence Theorem). Its range S is a compact convex set contained in $C_0 = \{a \in l^2 : |a_i| \leq \frac{1}{i+1}$ for all $i\}$. S has at least two points because $x^*(x) \neq x^*(y)$. Let $f_0 : S \to S$ be the map $h \circ f \circ h^{-1}$. In other words, if $a \in S$ we pick $z \in K$ such that $a = h(z)$ and define $f_0(a) = h(f(z))$. To see that this is well defined note that $a = h(z_1) = h(z_2)$ implies $x_i^*(z_1) = x_i^*(z_2)$ for all i which implies $x_i^*(f(z_1)) = x_i^*(f(z_2))$ for all i (because $B \leq B$) so $h(f(z_1)) = h(f(z_2))$ so f_0 is well defined. The fact that $B \leq B$ also implies that if $x_i^*(z_n) \to x_i^*(z)$ as $n \to \infty$ for each i then $x_i^*(f(z_n)) \to x_i^*(f(z))$ for each i. This means f_0 is continuous. [Convergence of a sequence in C_0 w.r.t. l^2 norm is equivalent to coordinatewise convergence]. Lemma 4 below shows that f_0 has a fixed point a.Let $K_1 = h^{-1}(\{a\})$. Let $z \in K_1$ so $h(z) = a$. Then $a = f_0(a) = h(f(z))$. Hence $f(z) \in K_1$. Thus $f(K_1) \subseteq K_1$. Clearly K_1 is convex. It is a closed subset of S and hence it is compact.

Lemma 5

Let $C_0 = \{a \in l^2 : |a_i| \leq \frac{1}{i+1}$ for all $i\}$. Then any continuous map $f : C_0 \to C_0$ has a fixed point.

Proof: let $A_n = \{(x_1, x_2, ..., x_n, 0, 0, ...) : x \in C_0\}$ and define $g_n : A_n \to A_n$ by $g_n(x) = (y_1, y_2, ..., y_n, 0, 0, ...)$ where $y = f(x_1, x_2, ..., x_n, 0, 0, ...)$. A_n can be identified with compact convex set in $\mathbb{R}^n$ and g_n is continuous, hence it has a fixed point $x^{(n)}$. [It is trivial to see that if every continuos map on a topological space X has a fixed point and Y is homeomorphic to X then every continuos map on Y has a fixed point]. Since $\{x^{(n)}\} \subseteq C_0$ and C_0 is compact in l^2 there is a subsequence $\{x^{(n_j)}\}$ converging to some $x \in C_0$. Let $y^{(n)} = f(x_1^{(n)}, x_2^{(n)}, ..., x_n^{(n)}, 0, 0, ...)$ so that $x^{(n)} = g_n(x^{(n)}) = (y_1^{(n)}, y_2^{(n)}, ..., y_n^{(n)}, 0, 0, ...)$. It is clear that $\{x_1^{(n)}, x_2^{(n)}, ..., x_n^{(n)}, 0, 0, ...\} \to x$ so $y^{(n)} \to f(x)$. Hence $x = \lim x^{(n_j)} = \lim(y_1^{(n_j)}, y_2^{(n_j)}, ..., y_n^{(n_j)}, 0, 0, ...)$
$= \lim y^{(n_j)} = f(x)$.

Lemma 6

If K is a closed convex subset of C_0 then every continuous map of K into itself has a fixed point.

Proof: this is similar to the proof of Theorem 2. For each $x \in C_0$ there is a unique point Px in K closest to x and the map $P : C_0 \to K$ is continuous. If $f : K \to K$ is continuous then $g : C_0 \to C_0$ defined by $g = f \circ P$ is continuous. Hence there exists $x \in C_0$ such that $f(P(x)) = x$. Since the range of f is contained in K we see that $x = f(P(x)) \in K$. But then $P(x) = x$ so $x = f(x)$.

Theorem 6 [Markov - Kakutani FPT]

Let K be a compact convex subset of a locally convex Hausdorff topological vector space X. Let $f_\alpha : K \to K$ ($\alpha \in I$) be a family of continuous functions that are affine (which means they satisfy the condition $f_\alpha(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f_\alpha(x_i)$ whenever $n \in \mathbb{N}$, $a_i \geq 0$ for all i and $\sum_{i=1}^{n} a_i = 1$). If $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all $\alpha, \beta \in I$ then there exists $x \in K$ such that $f_\alpha(x) = x$ for all $\alpha \in I$.

Proof: let $f_\alpha^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} f_\alpha^i$ where f_α^i is f_α composed with itself i times. $f_\alpha^{(n)}$ maps K into itself and any two members of $\{f_\alpha^{(n)} : \alpha \in I, n \geq 1\}$ commute. Let $\mathcal{F}$ be the collection of sets $f_\alpha^{(n)}(K), \alpha \in I, n \geq 1$. Each set in this collection is non-empty, compact and convex. This family also has finite intersection property: given $\alpha_j \in I, n_j \geq 1$ for $1 \leq j \leq N$ consider $(f_{\alpha_1}^{(n_1)} \circ f_{\alpha_2}^{(n_2)} ... \circ f_{\alpha_N}^{(n_N)})(K)$. It is clear that this non-empty set is contained in $(f_{\alpha_j}^{(n_j)})(K)$ for each j. Hence there is a point x_0 which belongs to $f_\alpha^{(n)}(K)$ for all $\alpha \in I, n \geq 1$. We claim that $f_\alpha(x_0) = x_0$ for all $\alpha \in I$ for all n. Since $x_0 \in f_\alpha^{(n)}(K)$ there exists $x \in K$ such that $x_0 = f_\alpha^{(n)}(x) = \frac{1}{n}[x + f_\alpha^1(x) + f_\alpha^2(x) + ... + f_\alpha^{n-1}(x)]$. Hence $f_\alpha(x_0) - x_0 = \frac{1}{n}[f_\alpha^1(x) + f_\alpha^2(x) + ... + f_\alpha^{n-1}(x) + f_\alpha^n(x)] - \frac{1}{n}[x + f_\alpha^1(x) + f_\alpha^2(x) + ... + f_\alpha^{n-1}(x)] = \frac{1}{n} f_\alpha^n(x) - \frac{1}{n} x \in \frac{1}{n}(K - K)$. This is true for each n and hence $f_\alpha(x_0) = x_0$. [Let U be a neighbourhood of 0. Then $K - K \subseteq mU$ for some m, since, otherwise, $\exists\, x_m \in K - K (m = 1, 2, ...)$ such that $\frac{1}{m} x_m \notin U$ for any m; by compactness of $K - K$ there is a subnet $\{x_{m_i}\}$ converging to some x. But then $\frac{1}{m_i} x_{m_i} \to 0$ contradicting the fact that $\frac{1}{m_i} x_{m_i} \notin U$ for any i. It now follows that $f_\alpha(x_0) - x_0 \in \frac{m}{m} U = U$. Since U is arbitrary and X is Hausdorff, $f_\alpha(x_0) = x_0$].

Notation: if p is a seminorm on X and $A \subseteq X$ we write $d_p(A)$ for $\sup\{p(x - y) : x, y \in A\}$.

Lemma 7

Let X be a Hausdorff locally convex topological vector space and K be non-empty, convex, weakly compact, separable subset. Let p be a weakly continuous semi-norm on X and $\varepsilon > 0$. Then there is a closed convex subset C of K such that $C \neq K$ and $p_d(K \backslash C) < \varepsilon$.

Proof: let $S = \{x \in X : p(x) \leq \varepsilon/4\}$. By Krein - Milman Theorem K has extreme points and it is the closed convex hull of the set of extreme points. Let D be the closure of the set of extreme points of K in the weak topology. Let $\{x_n\}$ be a countable dense subset of K. For each $x \in K$ the neighbourhood $\{y \in K : p(y - x) < \varepsilon/4\}$ must contain some x_n. Note that $x \in x_n + S$. Thus $K \subseteq \bigcup_n (x_n + S)$. Since S is weakly closed (because it is closed and convex) and D is a weakly compact subset of K there exists n_0 such that $(x_{n_0} + S) \cap D$ has non-empty interior in D. [D with the weak topology is a (locally) compact Hausdorff space so we can apply Baire Category Theorem].

Hence there exists a weakly open set U such that $U \cap D \subseteq (x_{n_0} + S) \cap D$ and $U \cap D \neq \emptyset$. Let K_1 be the closed convex hull of $D \backslash U$ and K_2 that of $D \cap U$. These two sets are weakly compact and convex. Any extreme point of K belongs to $D \subseteq (D|U) \cup (D \cap U) \subseteq K_1 \cup K_2$. Also $co(K_1 \cup K_2)$ is weakly closed (see below) and Krein - Milman Theorem implies $K \subseteq co(K_1 \cup K_2)$. But the reverse inclusion also holds so $K = co(K_1 \cup K_2)$.

Proof of the fact that $co(K_1 \cup K_2)$ is weakly closed: if $\alpha_i x_i + \beta_i y_i \to z$ then, through a subnet, $\alpha_i \to \alpha, \beta_i \to \beta, x_i \to x, y_i \to y$ and $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1, x \in K_1$ and $y \in K_2$. Hence $z = \alpha x + \beta y \in co(K_1 \cup K_2)$.

Next, we show that K_1 is a proper subset of K. Suppose $K_1 = K$, i.e., $K = \bar{co}(D \backslash U)$. This implies that the extreme points of K belong to $D \backslash U$. [All closures etc in this proof are w.r.t. weak topology. Suppose x_0 is an extreme point of K which does not belong to $D \backslash U$. There exists a continuous seminorm q (viz. Minkowski functional of a balanced convex neighbourhood V of 0 such that $x_0 + V$ does not intersect $D \backslash U$) such that $(D \backslash U) \cap \{x : q(x - x_0) < 1\} = \emptyset$. Let $U_0 = \{x : q(x) < 1/3\}$. Then $(x_0 + U_0) \cap ((D \backslash U) + U_0) = \emptyset$. Hence $x_0 \notin [(D \backslash U) + U_0)]^-$. Now $D \backslash U$ is compact. (We are referring to the weak topology). Hence $D \backslash U \subseteq \bigcup_{i=1}^{k} (y_i + U_0)$ for some finite set $\{y_1, y_2, ..., y_k\} \subseteq D \backslash U$. The closed convex hull H_i of $((D \backslash U) \cap (y_i + U_0))$ is contained in $y_i + \bar{U_0}$. Also $H_i \subseteq K$. As shown above $co(\bigcup_{i=1}^{k} H_i)$ is closed. Hence $K = [co(D \backslash U)]^- \subseteq co(\bigcup_{i=1}^{k} H_i)$. Since $x_0 \in K$ we can write x_0 as $\sum_{i=1}^{k} a_i x_i$ with $a_i's \geq 0$ $\sum_{i=1}^{k} a_i = 1$ and $x_i \in H_i$ for all i. Since x_0 is an extreme point of K it follows that $x_0 \in H_i$ for some i. Hence $x_0 \in y_i + \bar{U_0} \subseteq [D \backslash U + \bar{U_0}]^-$, a contradiction]. [We now switch back to the original topology of X]. Now since $D \backslash U$ is weakly closed (because U is weakly open) so $D \subseteq D \backslash U$ by the definition of D. This means $D \cap U = \emptyset$. This is a contradiction to the choice of U. We have proved that $K_1 \neq K$.

Recall that $U \cap D \subseteq (x_{n_0} + S) \cap D$. Hence K_2, the closed convex hull of $D \cap U$ is contained in $x_{n_0} + S = \{x : p(x - x_0) \leq \varepsilon/4\}$. Hence the diameter of $d_p(K_2) \leq \varepsilon/2$. For $0 < r \leq 1$ let $f_r(x_1, x_2, t) = tx_1 + (1 - t)x_2$ for $x_1 \in K_1, x_2 \in K_2, t \in [r, 1]$. Let C_r be the range of this function. Since f_r is continuous, C_r is weakly compact and convex. [Indeed, K_1 and K_2 are weakly close and f_r is (weak,weak,usual) to weak continuous. Convexity of C_r is proved as follows: $\alpha(tx_1 + (1-t)x_2) + (1-\alpha)((sy_1 + (1-s)y_2) = \beta z_1 + (1-\beta)z_2$ where $\beta = \alpha t + (1-\alpha)s, z_1 = \frac{\alpha t x_1 + (1-\alpha)s y_1}{\beta}, z_2 = \frac{\alpha(1-t)x_2 + (1-\alpha)(1-s)y_2}{1-\beta}$].

Claim: $C_r \neq K$ for any $r \in (0, 1]$. If $C_r = K$ and z is an extreme point of K then $z \in C_r$ so $z = tx_1 + (1 - t)x_2$ for some $x_1 \in K_1, x_2 \in K_2, t \in [r, 1]$. But then $t = 1$ or $z = x_1 = x_2$. In either case $z = x_1 \in K_1$ so $ext(K) \subseteq K_1$. This implies $K \subseteq K_1$ and hence $K = K_1$. This contradiction shows that $C_r \neq K$ for any $r \in (0, 1]$. Now note that $C_r \subseteq K$. Thus there exists a point z in $K\backslash C_r$. Since $K = co(K_1 \cup K_2)$ we can write z as $tx_1 + (1-t)x_2$ with $x_1 \in K_1, x_2 \in K_2, t \in [0,1]$. Since $z \notin C_r$ we must have $t < r$. Now $p(z-x_2) = p(tx_1 - tx_2) = tp(x_1 - x_2) \leq rd_p(K)$. Let $u = sy_1 + (1-s)y_2 \in K\backslash C_r$ ($y_1 \in K_1, y_2 \in K_2$,$0 < s < r$). Then $p(z-u) \leq p(z-x_2) + p(x_2-y_2) + p(y_2-u) \leq rd_p(K) + p_d(K_2) + rd_p(K)$ [since the argument used for $p(z-x_2) \leq rd_p(K)$ shows that $p(y_2-u) \leq rd_p(K)$]. Thus $p(z-u) \leq 2rd_p(K) + \varepsilon/2$. We have proved that $d_p(K\backslash C_r) \leq 2rd_p(K) + \varepsilon/2$. If $d_p(K) = 0$ we are done. Otherwise, take $r = \frac{\varepsilon}{4d_p(K)}$ to get $p_d(K\backslash C_r) \leq \varepsilon$.

Let $A \subseteq X$ and $\{T_i\}_{i \in I}$ a family of maps from A into itself. We say $\{T_i\}$ is a contracting family if $\exists\ x \neq y$ in A such that $0 \in \{T_i x - T_i y : i \in I\}^-$. The family is a NCF (non-contracting family) if it is not contracting.

Remark: any subfamily of a NCF is a NCF.

Lemma 8

With above notations $\{T_i\}$ is a NCF iff $x \neq y$ $(x, y \in A)$ implies $\exists$ a continuous seminorm p such that $\inf\{p(T_i(x) - T_i(y)) : i \in I\} > 0$.

Proof: suppose $x \neq y$ implies $\exists$ a continuous seminorm p such that $\inf\{p(T_i(x) - T_i(y)) : i \in I\} > 0$. If $x \neq y$ then $\{z : p(z) < \delta\}$ is a neighborhood of 0 which does not intersect $\{T_i x - T_i y : i \in I\}$ provided $0 < \delta < \inf\{p(T_i(x) - T_i(y)) : i \in I\} > 0$. Hence $0 \notin \{T_i x - T_i y : i \in I\}^-$ and the given family is not contracting. Conversely suppose $\{T_i\}$ is a NCF. Suppose $x \neq y$. Then $0 \notin \{T_i x - T_i y : i \in I\}^-$ and there is a balanced convex neighbourhood U of 0 such that $U \cap \{T_i x - T_i y : i \in I\} = \emptyset$. Let p be the Minkowski function al of U. Then p is a continuous seminorm and $p(T_i(x) - T_i(y)) \geq 1$ for all i.

Theorem 7 [Ryll -Nardzewski FPT]

Let X be a Hausdorff locally convex topological vector space and A be a weakly compact convex subset. Let $\{T_i : i \in I\}$ be a semigroup of affine maps each of which is weakly continuous. If this family is a NCF then it has a common fixed point in A.

Remarks: a family $\{T_i\}$ of maps on A is a semigroup if it is closed under composition. Any family generates a semigroup: just take all finite compositions of members of the family. We call this the semigroup generated by the given family.

Proof: we first prove that any finite subset $\{T_{i_1}, T_{i_2}, ..., T_{i_n}\}$ has a common fixed point. The map $S = \frac{T_{i_1} + T_{i_2} + ... + T_{i_n}}{n}$ is also affine and weakly continuous. By Markov - Kakutani FPT there exists x_0 such that $S(x_0) = x_0$. We claim that $T_{i_j} x_0 = x_0$ for all $j \leq n$. If $T_{i_j} x_0 \neq x_0$ for some j then we can rename the $T_{i_j}' s$ so that $T_{i_j} x_0 \neq x_0$ for $1 \leq j \leq m$ and $T_{i_j} x_0 = x_0$ for $m < j \leq n$. Let $W = \frac{T_{i_1} + T_{i_2} + ... + T_{i_m}}{m}$. Then $x_0 = Sx_0 = (\frac{T_{i_1} + T_{i_2} + ... + T_{i_m}}{n})x_0 + (\frac{T_{i_{m+1}} + T_{i_{m+2}} + ... + T_{i_n}}{n})x_0$

$= (\frac{T_{i_1}+T_{i_2}+...+T_{i_m}}{n})x_0 + \frac{n-m}{n}x_0$. This gives $Wx_0 = \frac{nx_0-(n-m)x_0}{m} = x_0$. We are now in a situation where, with obvious change of notations, $T_{i_j}(x_0) \neq x_0$ for any j but $S(x_0) = x_0$. In this case we proceed as follows: there exists $\varepsilon > 0$ and a continuos seminorm p such that for any T_l we have $p(T_l(T_{i_j}(x_0)) - T_l(x_0)) > \varepsilon$ (ε independent of l). [We used Lemma 6 with $T_{i_j}(x_0)$ and x_0 in place of x and y. The seminorm can depend on j but we can sum these seminorms over j]. Let $\mathcal{G}_1$ be the semigroup generated by $\{T_{i_j} : 1 \leq j \leq n\}$. Then $\mathcal{G}_1 \subseteq \{T_i\}$ and $\mathcal{G}_1$ consists of all finite compositions of the maps T_{i_j}. Hence $\mathcal{G}_1$ is a countable semigroup contained in $\{T_i\}$. Let K be the closed convex hull of $\{Tx_0 : T \in \mathcal{G}_1\}$. K is a separable weakly compact convex set and $K \subseteq A$. [It is weakly compact because it is a weakly closed subset of the weakly compact set A]. By Lemma 5 there exist a closed convex set $C \subseteq K$ such that $C \neq K$ and $p(K \backslash C) \leq \varepsilon$. For some $T \in \mathcal{G}_1$ we have $Tx_0 \notin C$. [For, $Tx_0 \in C$ for each $T \in \mathcal{G}_1$ implies $K \subseteq C$, so $K = C$, a contradiction]. Now $Tx_0 = TT_0(x_0) = \frac{TT_{i_1}x_0+TT_{i_2}x_0+...+TT_{i_n}x_0}{n}$ so $\frac{TT_{i_1}x_0+TT_{i_2}x_0+...+TT_{i_n}x_0}{n} \notin C$. It follows that $TT_{i_j}x_0 \notin C$ for some j. But $p_d(K \backslash C) \leq \varepsilon$ so $p(T(T_{i_j}x_0) - Tx_0) \leq \varepsilon$. This contradicts the choice of the seminorm p. We have proved that every finite subfamily of $\{T_i\}$ has a common fixed point. For any finite subfamily $\{T_{i_1}, T_{i_2}, ..., T_{i_n}\}$ of $\{T_i\}$ the set $\{x \in A : T_{i_j}(x) = x \text{ for } 1 \leq j \leq n\}$ is a weakly compact convex set. These compact sets have finite intersection property. Hence that is a point x in the intersection of these sets. x is a common fixed point for the family $\{T_i\}$.

Application to existence of Haar measure on compact groups.

Let Q be the set of all Borel probability measures on a compact topological group G. Give Q the weak* topology induced from $M(G) \equiv (C(G))^*$. Consider the following maps from Q into itself: $\mu \rightarrow_x \mu_y$ and $\mu \rightarrow [_x\mu_y]^{-1}$ where $(_x\mu_y)(E)\mu(xEy)$ and $\nu^{-1}(E) = \nu(E^{-1})$. These form a semigorup of affine maps on Q [See proof below]. If we show that this family is a non-contracting and that each of the maps in this semigroup is weakly continuous we can conclude that there is a probability measure P on G with $P(Ex) = P(yE) = P(E) = P(E^{-1})$ for all E Borel in G for all $x, y \in G$. Let us first show that above family is a semigroup. Write $L_x\mu(E) = \mu(xE), R_y\mu(E) = \mu(Ey), S\mu(E) = \mu(E^{-1})$. We are considering the family $\{SL_xR_y : x, y \in G\} \cup \{L_xR_y : x, y \in G\}$. We claim that $(SL_{x_1}R_{y_1})(SL_{x_2}R_{y_2}) = L_{y_1^{-1}x_2}R_{y_2x_1^{-1}}$. This would show easily that the family is a semigroup of affine maps. Now $(SL_{x_1}R_{y_1})(SL_{x_2}R_{y_2})\mu(E) = (SL_{x_1}R_{y_1})\mu S(x_2Ey_2)$

$= (SL_{x_1}R_{y_1})\mu(y_2^{-1}E^{-1}x_2^{-1}) = S\mu(x_1y_2^{-1}E^{-1}x_2^{-1}y_1) = \mu(y_1^{-1}x_2Ey_2x_1^{-1}) = (L_{y_1^{-1}x_2}R_{y_2x_1^{-1}})\mu(E)$. The fact that these maps are continuous for the weak*topology is clear. It remains only to show that $0 \notin cl(\{T\mu - T\nu\}_T)$ where T ranges over all the operators in our semigroup and μ and ν are distinct probability measures. The maps $(x, y) \rightarrow SL_xR_y$ and $(x, y) \rightarrow L_xR_y$ are continuous and hence their images are compact. Thus $cl(\{T\mu - T\nu\}_T) = \{T\mu - T\nu\}_T$ and $0 \notin \{T\mu - T\nu\}_T$ since $\int fdT\mu = \int fdT\nu$ for all $f \in C(G)$ implies $\int gd\mu = \int gd\nu$ for all g.